SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 6

Problem 1. Let G be a connected Lie group. Show that $Z(G) = \ker(\mathrm{Ad})$, where $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

Problem 2. Classify the 2-dimensional connected Lie subgroups of $\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$.

Problem 3. Let G be the group of transformations of $\mathbb{R}^2$ obtained by compositions of tranlsations and homotheties $x \mapsto \lambda x$ for $\lambda \in \mathbb{R}_+$.

(1) Show that G is center-free (ie, that $Z(G) = \{e\}$).
(2) Find vector fields on $\mathbb{R}^2$ generating the actions by homotheties and translations.
(3) Show that the vector fields of the previous part form the basis of a Lie algebra (ie, that they span a space closed under Lie brackets).
(4) Compute the adjoint representation of its Lie algebra in the basis of the previous part.
(5) Build a matrix group H isomorphic to G.

Non-Graded.

Problem 4. If G is a Lie group, let $\bar{P}$ denote the set of continuous paths $\gamma : [0,1] \to G$ such that $\gamma(0) = e$. Define a multiplication

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_2(2t), & t \in [0, 1/2] \\ \gamma_1(2t-1)\gamma_2(1), & t \in [1/2, 1] \end{cases}$$

Define a relation that $\gamma_1 \sim \gamma_2$ if and only if $(\gamma_1 * \gamma_2)(1) = e$, and $\gamma_1 * \gamma_2$ is trivial in $\pi_1(G, e)$. Show that $\sim$ is an equivalence relation, $*$ descends to $P = \bar{P}/\sim$ as a well-defined group operation, and that the map $\pi : \bar{P} \to G$ defined by $\pi(\gamma) = \gamma(1)$ is a group homomorphism. Furthermore, build a topology on P so that π is a covering map. (This shows that the universal cover of a Lie group is a Lie group!)

Problem 5. Find all 2-dimensional connected Lie subgroups of $SL(2, \mathbb{R})$ up to conjugacy.

Problem 6. * Find all 2-dimensional connected Lie subgroups of $SL(3, \mathbb{R})$ up to conjugacy.